ECON2228.05 Econometric Methods

Module 2
Binary Dependent Variable

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Qualitative Outcomes

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  - A researchers performs an analysis of socio-economic factors that affect whether a student graduates from college or drops out.

Basic Classification Problem

- Consider a qualitative variable Y that for every observation i takes a single value from a finite set of possible unordered values $\mathcal{C} = \{y_1, y_2, \ldots, y_C\}$.
  - Y = eye color, $\mathcal{C} = \{\text{brown}, \text{blue}, \text{green}\}$
  - Y = medical diagnosis, $\mathcal{C} = \{\text{stroke}, \text{drug overdose}, \text{epileptic seizure}\}$
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- Given a feature vector X and a qualitative response Y, the classification task is to build a function $C(X)$ that takes as input the feature vector X and predicts its value for Y, i.e. $C(X) \in \mathcal{C}$.
- In most cases we are interested in estimating the probabilities that Y belongs to a category in $\mathcal{C}$ given X, i.e.

$$\Pr(Y = y_c|X) \quad \forall y_c \in \mathcal{C}$$

Linear Probability Model: Problems

- Suppose for our medical condition classification we code

$$Y = \begin{cases} 1, & \text{if } \texttt{Stroke} \\ 2, & \text{if } \texttt{Drug Overdose} \\ 3, & \text{if } \texttt{Epileptic Seizure} \end{cases}$$

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- A different coding

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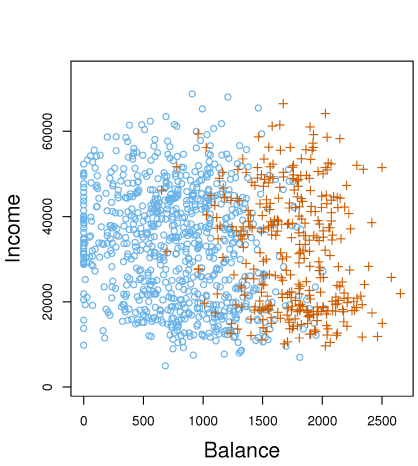
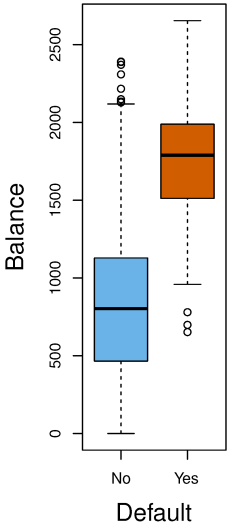
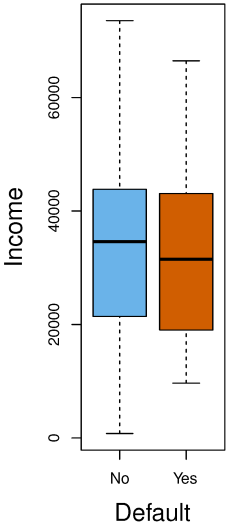
  Will generate a different model and different predictions

Example: Credit Card Defaults







Linear Probability Model

- Suppose for our credit card default classification we code

$$Y = \begin{cases} 0, & \text{if } \texttt{No} \\ 1, & \text{if } \texttt{Yes} \end{cases}$$

Can we use our standard regression model to estimate a regression of Y on X and classify outcome as `Yes` if $\widehat{Y} > 0.5$?

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- Given that Y is binary, we have

$$\mathbb{E}(Y|X) = 1 \cdot \Pr(Y = 1|X) + 0 \cdot \Pr(Y = 0|X) = \Pr(Y = 1|X)$$

  which means that in this case standard linear regression will estimate the probability of outcome $Y = 1$, hence the name *linear probability model* or *LPM*.

Linear Probability Model

- LPM retains all properties of linear regression, but the interpretation of the results is slightly different:

$$\Pr(Y = 1|X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_p X_p + \epsilon$$

$$\beta_j = \frac{\partial \mathbb{E}[\Pr(Y = 1|X)]}{\partial X_j} = \frac{\mathbb{E}[\Delta \Pr(Y = 1|X)]}{\Delta X_j}$$

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  so regression coefficients now capture constant *marginal probabilities* of outcome $Y = 1$ given a change in X_j.
- LPM works very well in binary classification, especially if sample size is moderate to large.
- But it also has some inherent disadvantages

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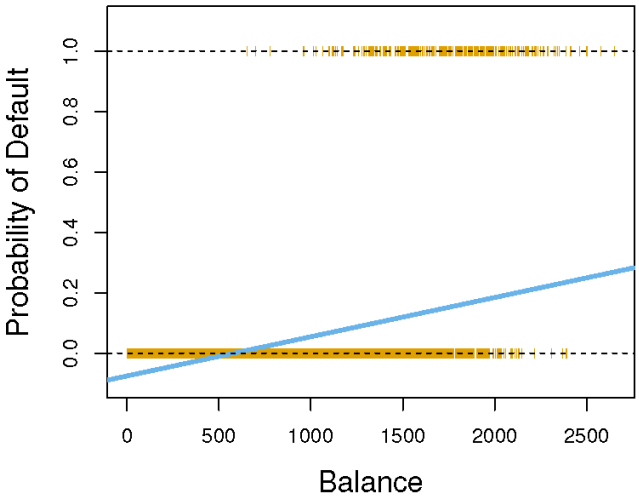

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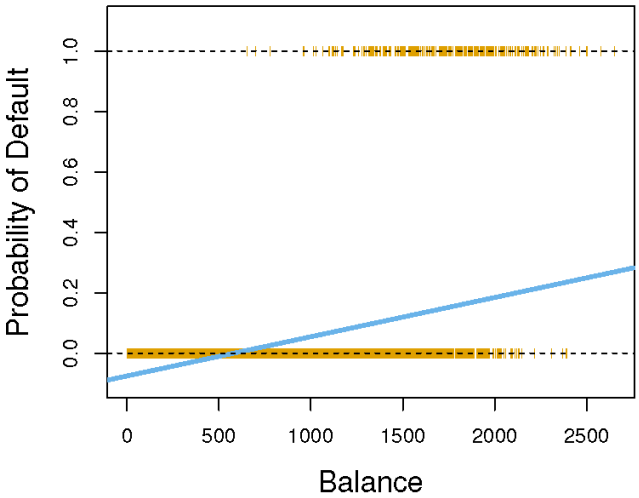

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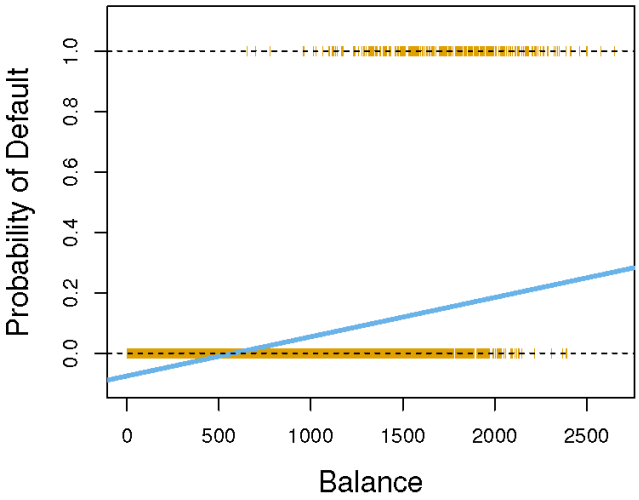

- The orange marks indicate the response Y, either 0 or 1.
- Blue line is estimated linear regression, which produces negative predictions for $\Pr(Y = 1|X)$ when `Balance` is less than 500.
- One way is to simply ignore the problem and bound any predictions from above and from below.
- But can we do better?

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$$\Pr(Y = 1|X) = F(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_p X_p)$$

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- For any random variable Z its cdf $F_Z(\cdot)$ is by definition:

$$F_Z(a) = \Pr(Z \leq a)$$

- In classical statistical learning the two most common choice for $F(\cdot)$ are standard normal and logistic cdf.

Probit and Logit Models

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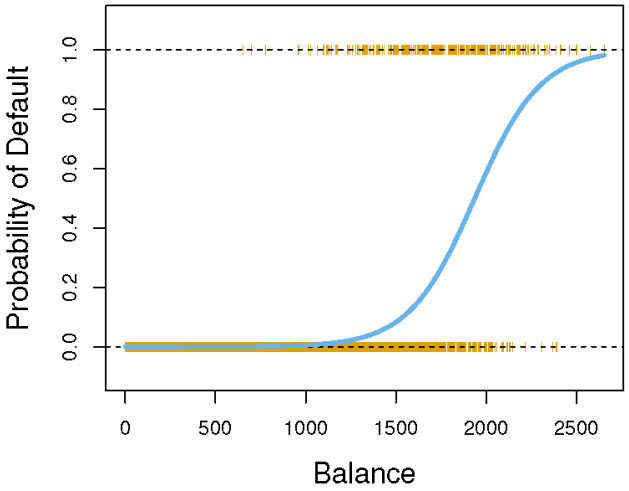
- In *logit* model $F(\cdot)$ is assumed to be the cdf of a logistic distribution:

$$F(\boldsymbol{X\beta}) = \Lambda(\boldsymbol{X\beta}) = \int_{-\infty}^{X\beta} \frac{\exp(-z)}{(1+\exp(-z))^2} dz = \frac{\exp^{X\beta}}{1+\exp^{X\beta}}$$

Probit and Logit Models



With either normal or logistic cdf as our $F(\cdot)$ function the estimated probability will, by definition, always lie in $[0, 1]$ interval.

Introduction to the Logit Model as a Latent Variable Model

- **Overview**: The logit model is commonly used in binary choice modeling, where the dependent variable takes values 0 or 1 (e.g., success or failure, yes or no). It can also be viewed as a latent variable model.
- **Latent Variable Framework**:
  - In the latent variable approach, the observed binary outcome is determined by an unobserved (latent) variable, which is modeled as a continuous variable.
  - The observed outcome is determined by whether this latent variable exceeds a certain threshold.
- **Key Idea**: The logit model can be interpreted as a probabilistic model where an individual's decision depends on the value of a latent variable (e.g., an underlying propensity to choose '1').

Latent Variable Representation in the Logit Model

- **Latent Variable Model**: The logit model assumes the existence of a latent variable y^* that is linearly related to the explanatory variables X and an error term ϵ:

$$y^* = X\beta + \epsilon$$

where:

  - y^* is the latent variable,
  - X represents the explanatory variables,
  - β is the vector of parameters,
  - ϵ is the error term, typically assumed to follow a logistic distribution.

- **Thresholding**: The observed binary outcome y is determined by whether the latent variable exceeds a certain threshold, often assumed to be 0:

$$y = \begin{cases} 1 & \text{if } y^* > 0 \\ 0 & \text{if } y^* \leq 0 \end{cases}$$

This thresholding leads to the discrete binary outcome.

Derivation of the Probability Function – Step 1

- **Step 1: Latent Variable Model**: As mentioned earlier, the latent variable model is:

$$y^* = X\beta + \epsilon$$

where $\epsilon \sim \text{Logistic}(0, 1)$, meaning the error term follows a logistic distribution with a mean of 0 and a variance of $\frac{\pi^2}{3}$.

- **Step 2: Define the Probability of $y = 1$**: The observed outcome y is related to the latent variable y^* through the following thresholding rule:

$$y = \begin{cases} 1 & \text{if } y^* > 0 \\ 0 & \text{if } y^* \leq 0 \end{cases}$$

The probability of observing $y = 1$ is:

$$P(y = 1 \mid X) = P(y^* > 0) = P(X\beta + \epsilon > 0)$$

which simplifies to:

$$P(y = 1 \mid X) = P(\epsilon > -X\beta) = F_\epsilon(X\beta)$$

where F_ϵ is the CDF of the logistic distribution.

Derivation of the Probability Function – Step 2

- **Step 3: Using the Logistic CDF**: The CDF of the logistic distribution is:

$$F(\epsilon) = \frac{1}{1 + \exp(-\epsilon)}$$

Therefore, the probability of $y = 1$ becomes:

$$P(y = 1 \mid X) = \frac{1}{1 + \exp(-X\beta)}$$

- **Step 4: Probability of $y = 0$**: The probability of $y = 0$ is simply:

$$P(y = 0 \mid X) = 1 - P(y = 1 \mid X) = \frac{\exp(-X\beta)}{1 + \exp(-X\beta)}$$

Thus, we have derived the probabilities for the logit model as:

$$P(y = 1 \mid X) = \frac{1}{1 + \exp(-X\beta)} \quad \text{and} \quad P(y = 0 \mid X) = \frac{\exp(-X\beta)}{1 + \exp(-X\beta)}$$

Interpretation of β as a Change in Log Odds

- **Log Odds Interpretation**: The **log odds** of $y = 1$ (i.e., the log of the odds of success) is:

$$\text{log-odds}(y = 1 \mid X) = \log \left(\underbrace{\frac{P(y = 1 \mid X)}{P(y = 0 \mid X)}}_{ODDS} \right)$$

  Substituting the probabilities:

$$\text{log-odds}(y = 1 \mid X) = X\beta$$

- **Change in Log Odds**: The coefficient β_j associated with an explanatory variable X_j represents the change in the log odds of $y = 1$ for a one-unit change in X_j, holding all other variables constant.
  - If $\beta_j > 0$, a one-unit increase in X_j increases the log odds of $y = 1$, making the outcome more likely.
  - If $\beta_j < 0$, a one-unit increase in X_j decreases the log odds of $y = 1$, making the outcome less likely.
- **Example**: If $\beta_j = 0.5$, then a one-unit increase in X_j increases the log odds of success (i.e., $y = 1$) by 0.5. This corresponds to multiplying the odds of success by $e^{0.5} \approx 1.65$, or an approximately 65% increase in the odds.

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- Suppose $x_1, x_2, ..., x_n$ form a random sample from a normal distribution for which the mean μ is unknown and variance σ^2 is known
- The likelihood function of μ is

$$L(\mu) = f_n(\boldsymbol{x}|\mu) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \tag{2}$$

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- $L(0.5) = 0.038$
- $L(0.52) = 0.058$
- $L(0.54) = 0.073$
- $L(0.56) = 0.081$
- $L(0.58) = 0.073$

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$$\ln L = 56 \ln(p) + 44 \ln(1-p) \tag{4}$$

$$\tag{5}$$

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- Step 2: Differentiate the log likelihood to find the optimal parameter

$$\frac{56}{p} - \frac{44}{(1-p)} = 0 \tag{6}$$

$$56(1-p) - 44p = 0 \tag{7}$$

$$p = \frac{56}{100} \tag{8}$$

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- In such cases we need to construct the overall likelihood surface using the individual likelihoods
- Each individual coin toss follows a Bernoulli distribution. Suppose $X = 1$ when heads and 0 otherwise

$$L(p; x) = p^x(1 - p)^{1-x}$$

Maximum Likelihood Estimation: Method

- **Method 2**: Remember the data is given to us

Observation	Outcome (x)	Likelihood of outcome
1	1	p
2	0	1-p
3	1	p
...		
99	0	1-p
100	1	p
Total	56	?

- What is the overall/joint likelihood of entries in the second column?

Maximum Likelihood Estimation: Method

- **Method 2**: Remember the data is given to us

Observation	Outcome (x)	Likelihood of outcome
1	1	p
2	0	1-p
3	1	p
...		
99	0	1-p
100	1	p
Total	56	?

- What is the overall/joint likelihood of entries in the second column?
- Each coin toss is independent

$$L(p) = p.p.p.p..(1-p)(1-p)...(1-p) \tag{9}$$

$$L(p) = p^{56}(1-p)^{44} \tag{10}$$

$$\ln L(p) = \ln(p^{56}(1-p)^{44}) \tag{11}$$

OLS as a special case of MLE

- Main assumption: The errors follow a normal distribution with mean 0 and variance σ^2

$$\epsilon \sim \mathcal{N}(0, \sigma^2)$$
$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$
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- The likelihood function of (β) is

$$L(\beta) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}}$$
$$\ln(L(\beta)) = \sum_{i=1}^{n} \ln(\frac{1}{\sigma\sqrt{2\pi}}) - \sum_{i=1}^{n} \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}$$

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- The first term does not depend on (β) and the second term has a constant σ^2 that we can bring outside the summation

$$\ln(L(\beta)) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

Maximum Likelihood Estimation

- Because we no longer have a direct connection between Y and our structural component $\boldsymbol{X}\boldsymbol{\beta}$, we need to specify our loss function in a different way. Using our link function, we can for every observation i wright down the probability of observing a certain value of Y_i given values of $\boldsymbol{X}_i$

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- For example, for a logit model we have:

$$\Pr(Y = Y_i|\boldsymbol{X}_i) = \left(\frac{\exp^{X_i\beta}}{1+\exp^{X_i\beta}}\right)^{Y_i}\left(1-\frac{\exp^{X_i\beta}}{1+\exp^{X_i\beta}}\right)^{1-Y_i}$$

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- With the default assumption of i.i.d. observations we can wright down the joint probability or *likelihood function* of seeing our sample:

$$\ell(\boldsymbol{\beta}) = \prod_{i=1}^{n}\Pr(Y = Y_i|\boldsymbol{X}_i)$$

Maximum Likelihood Estimation

- *Maximum likelihood* estimation (ML) is a method that chooses parameters $\boldsymbol{\beta}$ so as to minimize the loss function in form of the negative of the log likelihood function:

$$\widehat{\boldsymbol{\beta}}_{ML} = \underset{\boldsymbol{\beta}}{\text{argmin}} - \ln \ell(\boldsymbol{\beta})$$

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$$\widehat{\boldsymbol{\beta}}_{ML} = \underset{\boldsymbol{\beta}}{\text{argmin}} - \ln \ell(\boldsymbol{\beta})$$

- Under some general conditions $\widehat{\boldsymbol{\beta}}_{ML}$ is efficient, consistent and asymptotically normal, just like $\widehat{\boldsymbol{\beta}}_{OLS}$

	Coefficient	Std. error	z-statistic	p-value
`Intercept`	−10.6513	0.3612	−29.5	<0.0001
`balance`	0.0055	0.0002	24.9	<0.0001

- But unlike OLS, ML is a more general estimation procedure and allows one to recover structural parameters such as $\boldsymbol{\beta}$ in models that are far more flexible than standard MLR.

Logistic Regression

- A key difference of logit/probit models from LPM is the fact that margial effects are now calculated and interpreted in a different way:

$$\Pr(Y = 1|\boldsymbol{X}) = p(\boldsymbol{X}) = \frac{\exp^{X\beta}}{1 + \exp^{X\beta}}$$

$$\frac{p(\boldsymbol{X})}{1 - p(\boldsymbol{X})} = \exp^{X\beta}$$

$$\ln \frac{p(\boldsymbol{X})}{1 - p(\boldsymbol{X})} = X\beta$$

- The quantity on the left is called *log-odds* or *logit*
- The logistic regression model has a logit that is linear in X
- In a logistic regression model, increasing X by one unit changes the log odds by its corresponding β

Marginal Effects in Probit/Logit

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  where $p(\boldsymbol{X}) = \Pr(Y = 1|\boldsymbol{X})$ for simplicity
- The marginal effects now depend on values of all variables in $\boldsymbol{X}$, so we need to either estimate the marginal effects at a specific value of all our predictors (typically means or medians) or calculate their average over all values of $\boldsymbol{X}$ in our sample.

Multinomial Logistic Regression

- It is possible to extend the two-class logistic regression approach to the setting of $K > 2$ classes
- Suppose we have K classes. First we need to define a base class (K^{th} one)

$$\Pr(Y = k|\boldsymbol{X}) = \frac{\exp^{X\beta_k}}{1 + \sum_{l=1}^{K-1} \exp^{X\beta_l}}$$

$$\Pr(Y = K|\boldsymbol{X}) = \frac{1}{1 + \sum_{l=1}^{K-1} \exp^{X\beta_l}}$$

- It can be shown that the above model implies

$$\frac{\Pr(Y = k|\boldsymbol{X})}{\Pr(Y = K|\boldsymbol{X})} = \exp^{X\beta_k}$$

$$\ln \frac{\Pr(Y = k|\boldsymbol{X})}{\Pr(Y = K|\boldsymbol{X})} = X\beta_k$$

- The interpretation of the β_k is with respect to the base category K

Goodness-of-fit Measures

- While LPM can use the standard R^2 as a well-defined goodness-of-fit measure, it is not an option for logit/probit models due to the different loss function.
- In standard MLR an single R^2 value of 0.95 is an evidence of an excellent fit, but in classification problems we are often more interested in *class-specific* performance, especially in areas such as medicine or biology.

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- Consider the case where $Y = 1$ means a positive test result. Then our model's predictions fall into one of the 4 possible cases:

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  - For city administration FPs in traffic cameras and speeding tickets are more important.
  - In judicial system both FP and FN are equally important.

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- Consider the following *confusion matrix*, depicting the prediction results for the `Default` dataset, using 50% as a threshold for probability of default:

		True default status		
		No	Yes	Total
Predicted default status	No	9644	252	9896
	Yes	23	81	104
	Total	9667	333	10000

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  - However, out of 333 true defaults we managed to miss $252/333 = 75.67\%$, which could be an unacceptably high error rate for this class.

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- This why in classification problems it is important to evaluate class-specific precision via the following four measures:

	$Y = 0$	$Y = 1$
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- As one can easily see, all four measures are related to each other. In particular, the following two identities must always hold:

$$\text{TNR} + \text{FPR} = 100\% \quad \text{and} \quad \text{TPR} + \text{FNR} = 100\%$$

Goodness-of-fit Measures

The table on the previous slide was constructed using the rule $\widehat{Y} = 1$ if $\widehat{\Pr}(Y = 1|\boldsymbol{X}) > 0.5$, because 0.5 is the most common probability threshold used for classification predictions. However, the values of all 4 goodness-of-fit metrics will change if we change this threshold.

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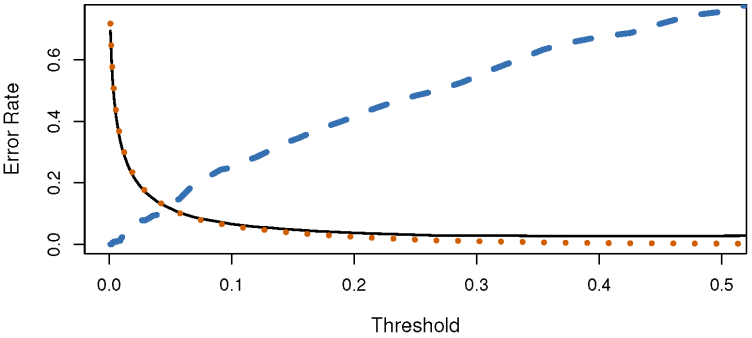

Black line: total error rate

Blue dashes: FNR

Orange dots: FPR

Based on this chart, we might want to set our threshold to 0.05 to achieve better error rate composition.

Deviance and Model Fit

- Another way of assessing the goodness of fit the Logit model is based on the concept of Deviance.
- Intuitively, deviance measures the discrepancy of the fitted logistic model with respect to a perfect model for $P[Y = 1 \mid X_1 = x_1, \ldots, X_k = x_k]$. This perfect model, known as the **saturated model**, represents an abstract model that fits the sample perfectly. That is, the model satisfies:

$$\hat{P}[Y = 1 \mid X_1 = X_{i1}, \ldots, X_k = X_{ik}] = Y_i, \quad i = 1, \ldots, n.$$

  The saturated model assigns probabilities 0 or 1 to Y depending on the actual value of Y_i.
- More precisely, the deviance is defined as the difference in likelihoods between the fitted model and the saturated model:

$$D = -2 \log \text{lik}(\hat{\beta}) + 2 \log \text{lik}(\text{saturated model}).$$

  Since the likelihood of the saturated model is exactly one, the deviance is simply another expression of the likelihood:

$$D = -2 \log \text{lik}(\hat{\beta}).$$

  As a consequence, the deviance is always greater than or equal to zero, and it is equal to zero only if the fit is perfect.

- A benchmark for evaluating the magnitude of the deviance is the null deviance:

$$D_0 = -2 \log \text{lik}(\hat{\beta}_0),$$

where $\hat{\beta}_0$ represents the parameters of the model with only an intercept (no predictors).

- The null deviance, D_0, represents the deviance of the worst model (the one fitted without any predictor) compared to the perfect model:

$$Y \mid (X_1 = x_1, \ldots, X_k = x_k) \sim \text{Bernoulli}(\text{logistic}(\beta_0)),$$

where the parameter $\hat{\beta}_0$ is given by:

$$\hat{\beta}_0 = \text{logit}(m/n) = \log \frac{m/n}{1 - m/n},$$

and m is the number of 1's in $Y_1, \ldots, Y_n$.

- The null deviance serves as a benchmark for comparing how much the model has improved by adding the predictors $X_1, \ldots, X_k$. This comparison can be quantified using the R^2 statistic, which is a generalization of the coefficient of determination in multiple linear regression:

$$Pseudo - R^2 = 1 - \frac{\text{deviance fitted model } (D)}{\text{deviance null model } (D_0)} = 1 - \frac{\log \text{lik}(\hat{\beta})}{\log \text{lik}(\hat{\beta}_0)}$$

  This global measure of fit shares some important properties with the R^2 in linear regression:
  - It is a quantity between 0 and 1.
  - If the fit is perfect, then $D = 0$ and $R^2 = 1$.
  - If the predictors do not add anything to the regression, then $D = D_0$ and $R^2 = 0$.
- Notice that this R^2 is not the percentage of variance explained by the logistic model, but rather a ratio indicating how close is the fit to being perfect or the worst.